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On Some Special Squared Rectangles

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A rectangle dissected into $N > 1$ non-overlapping squares is called a *squared rectangle* of *order* N. The N squares are the *elements* of the squared rectangle, but the term "element" is also used for the length of the sides of a square. A squared rectangle that does not contain a smaller rectangle is called *simple*. If the elements are all unequal, the squared rectangle is called *perfect*.

A table of simple squared rectangles of orders 9 to 15, inclusive, of which 3663 are perfect, is still available in a limited number of copies [1]. A much bigger table, of which only two copies exist [2], lists the 154490 simple perfect squared rectangles of orders 9 to 18, inclusive, according to non-decreasing ratio of shorter to longer side of rectangle, each squared rectangle being characterized by an appropriate numerical code. Special pairs of simple perfect squared rectangles, such as conformal rectangles (same side ratios) or congruent rectangles (same sides), are thus easy to obtain from this table.

In particular, we have looked for pairs of congruent rectangles that are dissected by the same set of squares although these squares are differently arranged in the two dissections. The pair of simple perfect squared rectangles of lowest order, $N = 13$, having this property is shown in Figure 1. It was constructed by Brooks, Smith, Stone, and Tutte, and its structure is well-understood from the theory of planar 3-connected graphs and electrical networks [3, Section 7.1].

As our table shows, such pairs do not exist for orders 14 and 15, and there is just one pair of order 16 (see Figure 2). Further, there are twelve pairs of order 17 and forty-five pairs of order 18, making a total of fifty-nine pairs of these special squared rectangles of orders less than 19.

Of course, drawing these fifty-nine pairs of rectangles would take too much space, not to mention that drawing them to scale is virtually impossible because of the large variations in size of the squares in the majority of cases. Instead, we shall list them by their numerical codes as displayed in the table. Here N denotes the order, V the vertical side, H the

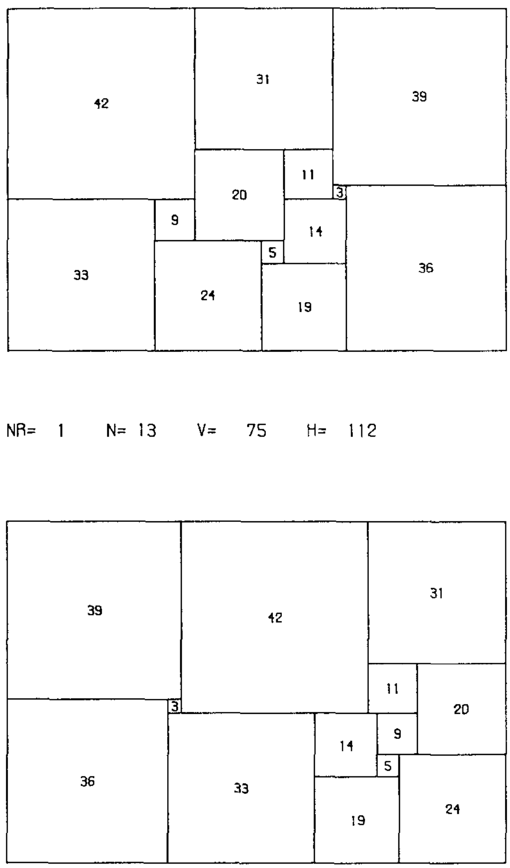

FIG. 1. Pair of simple perfect squared rectangles of lowest order ($N = 13$) dissected by the same set of squares in two different ways [3].

horizontal side of the rectangle. Moreover, given the value of H, each line in the column "CODES" is a unique representation of the corresponding squared rectangle: it is the ordered sequence of the elements encountered if we scan the squared rectangle from top left to bottom right, after having attached the size of each square to its top left corner. This code differs from the usual one in that parentheses are eliminated, so that the two codes of one pair are of equal length.

In Figure 3 we show the pair No. 57, which differs in structure from all others in the table in that the underlying electrical network has not two vertices at equal potential.

Finally, it should be remarked that the originals of our figures were automatically drawn by the computer/plotter.

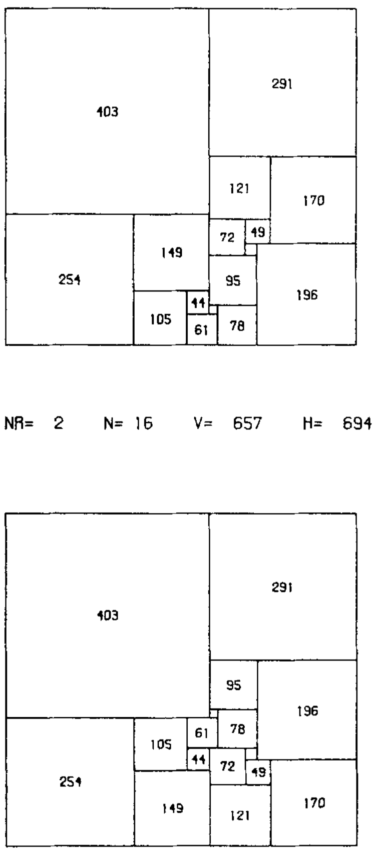

FIG. 2. Pair of simple perfect squared rectangles of order $N = 16$ dissected by the same set of squares in two different ways. The tiny squares of sides 17 and 23 were left unmarked.

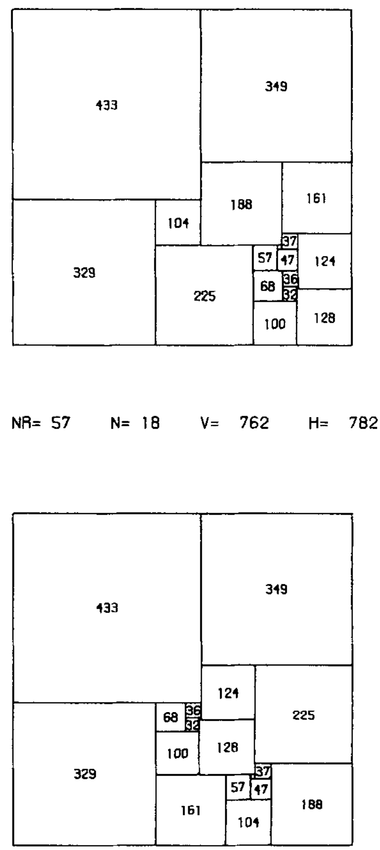

FIG. 3. Pair of simple perfect squared rectangles of order $N = 18$ dissected by the same set of squares in two different ways. The tiny squares of sides 4 and 10 were left unmarked. The structure is different from all others in the table.

TABLE I

TABLE OF SIMPLE PERFECT SQUARED RECTANGLES WHERE THE ELEMENTS CAN BE ARRANGED IN TWO DIFFERENT WAYS

	N	V	H	CODES
1	13	75	112	39,42,31,11,20,36,3,33,14,9,5,24,19 42,31,39,20,11,3,36,33,9,14,24,5,19
2	16	657	694	403,291,95,196,17,78,254,105,61,44,72,23,49,170,149,121 403,291,121,170,254,149,72,49,23,196,95,105,44,17,78,61
3	17	661	984	381,246,357,135,111,24,140,304,280,144,116,28,64,164,136,36,100 381,246,357,135,111,164,304,280,136,100,36,64,144,28,24,140,116
4	17	119	171	65,44,62,21,23,5,57,54,32,28,4,24,22,8,2,6,14 65,44,62,21,23,5,57,54,22,8,2,6,24,14,32,4,28
5	17	317	448	171,112,165,59,53,66,152,146,84,28,38,62,22,18,10,48,40 171,112,165,59,53,28,38,152,146,62,22,40,10,48,84,18,66
6	17	583	777	293,296,188,108,80,28,52,290,3,287,87,37,24,13,63,50,200 296,188,293,80,108,28,52,3,290,287,37,24,87,13,63,50,200
7	17	462	581	300,281,100,181,162,67,39,32,7,25,28,18,43,95,19,81,62 300,281,19,81,181,162,95,62,43,100,67,28,18,25,39,7,32
8	17	761	884	445,439,6,35,76,322,316,70,36,29,7,16,41,34,9,25,246 445,439,41,76,322,316,70,34,25,9,16,36,7,6,35,29,246
9	17	373	432	197,123,112,39,73,95,28,67,33,40,176,21,155,54,7,47,101 197,112,123,73,39,28,95,67,33,40,176,21,54,7,155,47,101
10	17	709	821	412,409,33,76,300,297,73,42,14,19,31,11,9,5,24,20,224 412,409,14,19,76,300,297,73,31,11,20,5,24,42,9,33,224
11	17	490	562	289,273,16,40,217,201,57,26,21,5,13,3,10,33,31,23,144 289,273,23,33,217,201,57,31,13,10,3,40,16,26,5,21,144
12	17	490	553	293,260,30,230,3,8,19,197,71,16,7,5,2,11,9,55,126 293,260,30,230,11,19,197,71,16,9,3,8,7,2,5,55,126
13	17	956	1024	577,447,109,129,209,89,20,69,80,379,198,147,11,300,181,17,164 577,447,147,300,379,181,17,164,11,80,209,198,89,69,20,129,109
14	17	504	539	284,255,129,126,220,64,28,36,20,8,3,123,12,44,120,32,76 284,255,129,126,220,64,36,28,8,20,3,123,44,12,120,32,76
15	18	118	185	68,69,48,21,27,50,17,1,16,49,20,6,14,19,33,29,5,24 69,48,68,27,21,1,17,50,49,20,6,16,14,19,33,29,5,24
16	18	244	370	137,120,113,52,61,47,73,107,30,43,9,77,70,51,22,7,36,29 137,120,113,7,36,70,47,51,29,107,30,22,43,77,73,9,61,52
17	18	357	528	185,198,145,53,92,172,13,159,37,29,39,8,21,45,11,120,32,77 198,145,185,92,53,13,172,159,39,29,37,21,8,45,120,11,32,77
18	18	1121	1610	579,616,415,201,214,542,37,505,238,80,31,18,196,49,129,29,296,267 616,415,579,214,201,37,542,505,80,31,238,18,196,49,129,296,29,267
19	18	735	1043	450,272,321,223,49,174,196,285,165,135,240,22,218,120,45,30,105,75 450,272,321,223,49,174,196,285,120,45,75,105,240,22,218,165,30,135
20	18	1265	1784	697,446,641,251,195,56,156,624,568,336,100,256,232,84,20,64,212,148 697,446,641,251,195,212,624,568,232,148,84,64,56,156,20,100,336,256
21	18	503	805	305,312,188,76,112,48,28,20,8,33,87,298,7,291,75,21,54,216 312,188,305,112,76,28,48,8,20,33,87,7,298,291,21,75,54,216
22	18	1221	1524	640,400,584,116,100,184,16,84,64,68,60,4,56,231,637,581,175,406 640,400,584,64,68,84,184,60,4,56,16,100,116,231,637,581,175,406
23	18	207	273	112,65,96,36,14,15,13,1,16,9,4,5,111,11,39,95,28,67 112,65,96,36,13,16,9,4,1,15,5,14,111,11,39,95,28,67
24	18	1361	1784	704,400,680,164,100,136,64,36,28,144,140,116,24,235,681,657,211,446 704,400,680,140,116,144,24,64,28,164,36,136,100,235,681,657,211,446
25	18	1159	1496	604,300,592,96,80,124,36,44,76,20,56,168,132,193,567,555,181,374 604,300,592,132,168,96,36,80,124,76,20,56,44,193,567,555,181,374
26	18	1345	1704	741,426,537,315,111,204,444,604,336,320,80,364,16,100,284,268,84,184 741,426,537,315,111,284,364,604,268,184,204,80,444,84,100,336,16,320
27	18	1211	1520	663,441,416,129,287,337,104,233,548,115,75,212,433,190,137,53,296,243 663,416,441,287,129,104,337,233,548,115,75,212,190,137,433,53.296,243
28	18	1271	1574	790,784,6,291,487,481,315,95,196,166,84,65,19,40,101,82,21,61 790,784,101,196,487,481,166,82,61,21,40,64,19,6,95,65,315,291
29	18	1235	1496	639,374,483,265,109,256,336,596,308,176,80,416,52,124,288,20,72,196 639,374,483,265,109,176,416,596,288,20,196,72,124,80,336,308,52,256

TABLE I (*continued*)

TABLE OF SIMPLE PERFECT SQUARED RECTANGLES WHERE THE ELEMENTS CAN BE ARRANGED IN TWO DIFFERENT WAYS
(CONTINUED)

30	18	1331	1610	696,476,438,119,319,169,226,81,200,112,57,635,61,228,55,173,574,401 696,438,476,319,119,81,169,226,200,112,57,635,61,55,228,574,173,401
31	18	1206	1454	743,711,216,495,463,280,89,127,51,38,183,97,13,152,64,86,11,75 743,711,89,127,495,463,183,97,51,38,13,152,86,11,75,280,64,216
32	18	643	765	341,216,208,60,148,164,52,112,302,39,24,37,87,263,63,13,50,200 341,208,216,148,60,52,164,112,302,39,24,87,37,63,13,263,50,200
33	18	1441	1704	724,284,160,196,340,124,36,88,144,440,56,540,717,291,156,135,561,426 724,440,540,284,156,56,144,340,124,88,36,196,160,717,291,135,561,426
34	18	1404	1645	739,444,462,315,129,111,351,240,665,74,59,256,69,5,591,64,133,389 739,462,444,129,315,351,111,240,665,74,59,256,591,69,5,64,133,389
35	18	1413	1636	825,811,14,39,156,602,588,142,84,25,64,58,21,5,16,53,37,446 825,811,53,156,602,588,142,58,37,21,16,14,39,5,25,84,64,446
36	18	1321	1524	769,755,14,15,160,566,552,146,46,21,17,1,16,4,29,25,100,406 769,755,29,160,566,552,146,46,25,14,15,21,4,17,1,16,100,406
37	18	1461	1684	845,839,6,35,176,622,616,170,36,29,7,16,41,34,9,25,100,446 845,839,41,176,622,616,170,34,25,9,16,36,7,6,35,29,100,446
38	18	1234	1421	712,709,33,151,525,522,148,42,14,19,31,11,9,5,24,20,75,374 712,709,14,19,151,525,522,148,31,11,20,5,24,42,9,33,75,374
39	18	1418	1631	841,790,51,111,628,577,151,84,80,20,91,4,25,71,67,21,46,426 841,790,71,91,628,577,151,67,46,51,20,111,21,25,84,4,80,426
40	18	1272	1459	749,710,64,84,562,523,149,77,44,20,104,13,31,72,5,18,49,374 749,710,44,104,562,523,149,72,5,49,18,31,20,84,77,13,64,374
41	18	1466	1679	859,820,39,135,646,607,181,110,14,36,85,71,31,22,9,49,40,426 859,820,40,49,85,646,607,181,71,31,9,22,36,39,14,135,110,426
42	18	833	945	489,456,30,49,377,11,19,344,120,16,9,3,8,7,2,5,104,224 489,456,30,49,377,3,8,19,344,120,16,7,5,2,11,9,104,224
43	18	938	1057	538,519,19,81,419,400,95,62,43,100,67,28,18,25,39,7,32,238 538,519,100,419,400,67,39,32,7,25,28,18,43,95,19,81,62,238
44	18	938	1057	557,500,62,438,381,143,28,5,24,43,20,7,1,6,19,13,95,238 557,500,62,438,381,143,20,13,19,43,7,6,1,5,28,24,95,238
45	18	167	185	98,87,23,36,28,69,29,10,13,8,20,7,3,4,44,12,40,32 98,87,23,28,36,69,29,10,13,20,8,7,3,4,12,44,40,32
46	18	952	1043	395,268,380,156,112,72,172,248,238,157,128,28,100,81,400,76,324,319 400,324,319,81,238,248,76,128,100,172,157,28,72,156,395,112,380,268
47	18	952	1043	572,471,233,238,380,80,112,52,28,24,4,96,20,76,248,5,243,172 572,471,233,238,380,112,80,28,52,4,24,96,20,76,248,5,243,172
48	18	1220	1326	753,573,268,305,467,286,231,37,342,181,105,83,148,76,29,18,65,47 753,573,231,342,467,181,105,83,148,76,29,47,65,37,305,286,18,268
49	18	236	254	137,117,20,43,54,99,58,32,11,65,13,19,41,17,7,6,25,24 137,117,20,13,19,65,7,6,25,99,41,24,17,32,58,11,54,43
50	18	633	670	342,328,55,121,142,291,51,60,56,4,68,84,21,64,163,116,16,100 342,328,55,121,142,291,51,116,100,21,163,60,56,16,84,4,68,64
51	18	261	275	150,125,54,61,111,39,34,27,72,31,7,20,41,18,13,5,28,23 150,125,54,61,111,39,27,34,72,31,20,7,18,13,41,5,28,23
52	18	554	581	317,264,133,66,65,237,80,1,64,67,4,60,157,71,56,15,101,86 317,264,133,65,66,237,80,64,1,67,60,4,157,56,71,101,15,86
53	18	541	564	309,255,53,28,24,41,109,7,17,25,3,10,68,78,232,77,177,155 309,255,78,68,109,232,77,10,17,41,53,28,7,24,25,3,177,155
54	18	469	483	243,240,11,229,106,92,36,9,3,8,7,2,5,20,56,14,134,120 243,240,3,8,229,106,92,36,7,5,2,11,9,20,56,14,134,120
55	18	469	483	270,213,39,30,44,100,16,14,32,7,2,56,25,199,71,57,156,128 270,213,57,156,199,128,56,100,71,32,25,7,16,2,14,44,39,30
56	18	1404	1441	799,642,291,351,605,194,231,60,105,89,411,16,73,121,64,9,240,185 799,642,351,291,605,194,60,231,411,105,89,16,73,121,64,9,240,185
57	18	762	782	433,349,124,225,329,68,36,32,4,128,100,37,188,161,57,10,47,104 433,349,188,161,329,104,37,124,225,57,10,47,68,36,32,4,128,100
58	18	644	659	366,293,132,161,278,88,33,70,29,84,4,190,37,43,64,106,21,85 366,293,161,132,278,88,29,33,70,190,84,4,37,43,64,106,21,85
59	18	441	448	240,208,32,14,10,16,38,98,4,6,18,22,201,89,60,23,135,112 240,208,32,18,22,38,98,14,4,10,16,201,89,6,60,23,135,112

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